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427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, Ohio.

Of all ellipses circumscribed about a given parallelogram, the minimum [maximum], with regard to area, has as conjugate diameters the diagonals of the parallelogram.

SOLUTION BY HENRY D. THOMPSON, Princeton, N. J.

Let the given parallelogram be $PQP'Q'$, with $QP = 2h$, and $Q'P = 2k$, let the diagonals $P'P$, $Q'Q$ intersect at O, and through O take the x-axis parallel to, and positive in, the direction of QP and $P'Q'$, and the oblique y-axis parallel to, and positive in, the direction $Q'P$ and $P'Q$, and let the angle of the axes be ω. Take a representative ellipse through $PQP'Q'$ cutting the positive x-axis in the point U, with the coördinates (u, o), and the y-axis in the point $V(o, v)$. Then the equation of this ellipse is

$$\frac{x^2}{u^2} + \frac{y^2}{v^2} = 1, \tag{1}$$

and since it passes through $P(h, k)$ the relation holds

$$\frac{h^2}{u^2} + \frac{k^2}{v^2} = 1. \tag{2}$$

The area for the ellipse is

$$uv\pi \sin \omega, \tag{3}$$

and this is a minimum when

$$M = uv \tag{4}$$

is a minimum. Differentiation of (4) and (2) gives for the minimum $dM/du = v + udv/du = 0$, with $-h^2/u^3 - (k^2/v^3)dv/du = 0$, and elimination of dv/du gives $k^2/v^2 = h^2/u^2 = \frac{1}{2}$ from (2), whence $u = h\sqrt{2}$, $v = k\sqrt{2}$. These values set in d^2M/du^2 give $4k/h$, a positive magnitude, and the area of the ellipse is a minimum. This ellipse is $x^2/2h^2 + y^2/2k^2 = 1$, and the tangents at the vertices evidently intersect on the diagonals $P'P$ and $Q'Q$, which are thus conjugate, since OU and OV are conjugate.

The theorem can be proved without the use of the calculus as follows: The coördinates of a point on (1) can be expressed by means of a parameter φ by the equations $x = u \cos \varphi$, $y = v \sin \varphi$ which for the point $P(h, k)$ become $h = u \cos \varphi_1$, $k = v \sin \varphi_1$, where φ_1 changes with u and v. Then (4) becomes $M = hk/\sin \varphi_1 \cos \varphi_1 = 2hk/\sin 2\varphi_1$, which has the single minimum when $\sin 2\varphi_1$ is a maximum, that is unity, or when $\varphi_1 = \pi/4$, or when $u = h\sqrt{2}$, $v = k\sqrt{2}$, as before.

Also solved by MARIE WHELAN, J. B. REYNOLDS, and FLORENCE P. LEWIS.

Several solutions of Mechanics problem 339 will be printed in an early issue. EDITORS.

340 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A rigid straight line l passes through a fixed point O, but is otherwise free to move in a plane. If C is the instantaneous center of rotation for l, prove that CO is always perpendicular to l and that, if (O being used as pole) $\rho = f(\theta)$ represents the locus of any point P on l, OC is always equal to $(d/d\theta)f(\theta)$.

SOLUTION BY S. W. REAVES, University of Oklahoma.

If we know the directions of motion at a given instant of two points of the moving line, then the center of rotation for that instant is to be found at the intersection of the normals drawn at these points to their directions of motion. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.)

In the problem before us the direction of motion of the point of l, which at the given instant coincides with O, is along l. Hence the normal to l at O must pass through C.

Again, the normal at P to the path of P, $\rho = f(\theta)$, must also pass through C. Hence the angle OPC is the complement of the angle ψ between OP and the tangent at P. Hence,

$$\tan OPC = \cot \psi = \frac{d\rho}{\rho d\theta}.$$

(See any book on Calculus.) Therefore, from the right triangle COP, we have

$$OC = \rho \tan OPC = \frac{d\rho}{d\theta} = \frac{d}{d\theta} f(\theta).$$

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

257 (Number Theory). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to 10^t, inclusive, every one of which contains the figure 9 exactly r times $(0 \leqq r \leqq t)$.

SOLUTION BY THE PROPOSER.

In the case of the integers from 1 to 10, we have nine which do not contain the figure 9 and one which contains one 9. This shall be indicated by the expression $9 + 1$.

In the case of 10^2, the number of integers, which do not contain 9, is $9 \cdot 9$, or 9^2; which contain one 9, is $9 \cdot 1 + 9$, or $2 \cdot 9$; which contain two 9's, is 1, and we have the expansion of

$$(9 + 1)^2 = 9^2 + 2 \cdot 9 + 1.$$

For 10^3, we have $9 \cdot 9^2$, $9 \cdot 2 \cdot 9 + 9^2$, $9 \cdot 1 + 2 \cdot 9$, and 1, or $9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$.

Then, for 10^k, assume the expansion of $(9 + 1)^k$, or

$$9^k + \binom{k}{1} 9^{k-1} + \binom{k}{2} 9^{k-2} + \cdots + \binom{k}{n-1} 9^{k-(n-1)} + \binom{k}{n} 9^{k-n} + \cdots + \binom{k}{k-1} 9 + 1.$$

For 10^{k+1} we reason as follows: The number of integers which do not contain 9 is $9 \cdot 9^k$, or 9^{k+1}; which contain one 9, is $9 \cdot \binom{k}{1} 9^{k-1} + 9^k$, or $\binom{k+1}{1} 9^k$; which contain two 9's, is $9 \cdot \binom{k}{2} 9^{k-2} + \binom{k}{1} 9^{k-1}$ or $\binom{k+1}{2} 9^{k-1}$, and which contain n 9's, is

$$9 \cdot \binom{k}{n} 9^{k-n} + \binom{k}{n-1} 9^{k-(n-1)} = \left[\binom{k}{n} + \binom{k}{n-1}\right] 9^{k-n+1} = \binom{k+1}{n} 9^{k+1-n}.$$

Hence, we have, for 10^{k+1}, the expansion of $(9 + 1)^{k+1}$, or

$$9^{k+1} + \binom{k+1}{1} 9^k + \cdots + \binom{k+1}{n} 9^{k+1-n} + \cdots + \binom{k+1}{k} 9 + 1.$$

Now, the derived expression holds for $k = 2$ and for $k = 3$; hence it holds for all positive integral values of k.

Therefore, the general expression required is $\binom{t}{r} 9^{t-r}$.

Also solved by HORACE OLSON, H. C. FEEMSTER, C. C. YEN, and N. P. PANDYA.

258 (Number Theory). Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for a_n, where the a's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer n.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Though this solution does not directly involve binomial coefficients, yet by finding the value of a_n it may be considered to dispose of the problem sufficiently.